WAVE EQUATION FOR FREE-SPACE OR LOSSLESS OR NON-CONDUCTING MEDIUM.

Consider an ~~wave~~ electromagnetic wave propagating in free space or in non-conducting or loss-less or perfect dielectric medium in which there is no conduction current ($J=0$) and no charge existed (i.e. $\rho=0$). Then the Maxwells equation becomes

$$\nabla \times H = \frac{\partial D}{\partial t} = \epsilon \frac{\partial E}{\partial t} \quad \text{—(I)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t} \quad \text{—(II)}$$

$$\nabla \cdot D = 0, \quad \nabla \cdot E = 0 \quad \text{—(III)}$$

$$\nabla \cdot B = 0, \quad \nabla \cdot H = 0 \quad \text{—(IV)}$$

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Now, differentiating equation (I) w.r.t time t. we get

$$\frac{\partial}{\partial t}(\nabla \times H) = \frac{\partial}{\partial t}\left(\frac{\partial D}{\partial t}\right) \quad \text{from (I)}$$

$$\frac{\partial}{\partial t}(\nabla \times H) = \frac{\partial^2 D}{\partial t^2}$$

or, $$\nabla \times \frac{\partial H}{\partial t} = \frac{\partial^2 D}{\partial t^2}$$

$$\nabla \times \frac{\partial H}{\partial t} = \mu\frac{\partial^2 H}{\partial t^2} \; \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad \text{—— (V)}$$

Now, taking curl on the both sides of eqn (II) we get

$$\therefore \nabla \times E = \frac{-\partial B}{\partial t}$$

$$\cancel{\nabla \times \nabla \times E}$$

$$\nabla \times \nabla \times E = \nabla \times \frac{-\partial B}{\partial t}$$

$$\nabla \times \nabla \times E = -\nabla \times \frac{\partial B}{\partial t}$$

$$= -\mu_0\left(\nabla \times \frac{\partial H}{\partial t}\right)$$

Now, putting the value of $\nabla \times \frac{\partial H}{\partial t}$ from equation (V) in above equation we get. $[\because B = \mu H]$

$$\nabla \times \nabla \times E = -\mu_0\left(\epsilon_0 \frac{\partial^2 E}{\partial t^2}\right)$$

$$\nabla \times \nabla \times E = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad \text{—— (VI)}$$

Now, $\because$ we know that,

$$\nabla \times \nabla \times E = (\nabla \cdot E)\nabla - \nabla^2 E$$

$$\therefore (\nabla \cdot E)\cdot\nabla - \nabla^2 E = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$0 \; \cancel{-}\, \nabla^2 E = \cancel{-}\, \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\boxed{\therefore \quad \nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}} \quad \text{——} (a)$$

This is the general wave equation in terms of Electric field intensity E, in similar way above wave equation for magnetic field intensity is given by

$$\boxed{\nabla^2 H = \mu_0 \epsilon_0 \frac{\partial^2 H}{\partial t^2}} \quad \text{——} (b)$$

Now, multiplying equation (a) by ϵ_0 and (b) by μ_0 we get

$$\nabla^2 (\epsilon_0 E) = \mu_0 \epsilon_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 (\epsilon_0 E) = \mu_0 \epsilon_0 \frac{\partial^2 (\epsilon_0 E)}{\partial t^2}$$

$$\boxed{\nabla^2 D = \mu_0 \epsilon_0 \frac{\partial^2 D}{\partial t^2}} \quad \text{——} (c)$$

$[\because D = \epsilon_0 E]$

Similarly $[\because B = \mu_0 H]$

$$\boxed{\nabla^2 B = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}} \quad \text{——} (d)$$

The above two equations are vector homogeneous wave equation for free space in terms of electric flux density (D) and magnetic flux density (B)

The equations ((a) and (b)) can also be represented in rectangular components form as

$$\nabla^2 E_x = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

$$\nabla^2 E_y = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$\nabla^2 E_z = \mu_0 \epsilon_0 \frac{\partial^2 E_z}{\partial t^2}$$

In terms of homogeneous vector wave equation in complex time harmonic form for free space (or vacuum) is given by

$$\boxed{\begin{aligned} \nabla^2 E &= -\omega^2 \mu_0 \epsilon_0 E \\ \nabla^2 H &= -\omega^2 \mu_0 \epsilon_0 H \end{aligned}}$$

The standard partial wave equation frequency encountered in mechanical engineering is given by

$$\nabla^2 X = \frac{1}{V^2} \frac{\partial^2 X}{\partial t^2}$$

where X is any desired field vector.
V = velocity of wave.

On comparing the above equation with equation (a) and (b) we get

$$V = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$\because$ $\mu_0 = 4\pi \times 10^{-7}$ F/m permeability of free space

$\epsilon_0 = 8.85 \times 10^{-12} = \frac{1}{36\pi} \times 10^{-9}$ H/m = permittivity of free space.

$$\therefore \quad V = \frac{1}{\sqrt{4\pi \cdot \frac{1}{36\pi} \times 10^{-7} \times 10^{-12}}} = 3 \times 10^8 \text{ m/s}$$

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This is the velocity of wave with which it travel in free space.